Homework Assignments

Bifurcations: Theory and Applications

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 9: Consider the parameter dependent iteration of the local C^∞ diffeomorphism, F, given by

$$x_{n+1} = F(\lambda, x_n), \qquad (\lambda, x) \in \mathbb{R}^k \times \mathbb{R}^N, \tag{1}$$

Assume a trivial fixed point, $F(\lambda, 0) \equiv 0$, $B \coloneqq D_x F(0,0)$. Thus, the linearization at the origin of the extended system

$$\begin{array}{rcl} x_{n+1} & = & F(\lambda_n, x_n)\,, \\ \lambda_{n+1} & = & \lambda_n\,, \end{array} \quad \text{is} \quad \tilde{B} \coloneqq \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{\mathbb{R}^k} \end{array} \right).$$

Compare the $\tilde{B}^T$ normal form of the full system with the B^T normal form of

$$x_{n+1} = F(0, x_n). \tag{2}$$

Prove that replacing the coefficients of the normal form to (2) by *properly chosen* polynomials in λ yields the normal form to (1).

Problem 10: Consider the parameter dependent iteration of the local C^∞ diffeomorphism, F, given by

$$x_{n+1} = F(\lambda, x_n) = (-1-\lambda)x_n + r(\lambda, x_n), \qquad \lambda, x_n \in \mathbb{R}.$$

Here, for fixed λ, $r(\lambda, x)$ is of order x^2.

(i) Show that, for λ small, the $B(\lambda)^T$ normal form

$$y_{n+1} := (-1-\lambda)y_n + \alpha_3(\lambda)y_n^3 + \dots$$

is odd, to any finite order.

(ii) Assume $\alpha_3(0) > 0$. Show that the iterations

$$(1)\ y_{n+1} = (-1-\lambda)y_n + \alpha_3(\lambda)y_n^3 + \dots \qquad \text{and} \qquad (2)\ \tilde{y}_{n+1} = (-1-\lambda)\tilde{y}_n + \tilde{y}_n^3 + \dots,$$

are (locally) topologically equivalent for λ fixed, small enough. In other words, show that there exists a local homeomorphism h such that if $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ solves (2), then

$$\{y_n\}_{n\in\mathbb{N}} := \{h(\tilde{y}_n)\}_{n\in\mathbb{N}}$$

solves (1). Seek h of the form $h(\tilde{y}) = \tilde{y} + h_2(\lambda)\tilde{y}^2 + h_3(\lambda)\tilde{y}^3 + \dots$

(iii) Study the stability of the trivial branch of fixed points $\tilde{y}_n \equiv 0$ depending on λ and show that if $\lambda > 0$ small, then the truncated iteration

$$\tilde{y}_{n+1} = (-1-\lambda)\tilde{y}_n + \tilde{y}_n^3,$$

has period 2 solutions near $\tilde{y}_n \equiv 0$.

Problem 11: Show that the Lie group $SO(2)$ is path connected, but not simply connected. How about $SU(2)$?

Hint: Prove first that the groups consist of matrices of the form

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

for α, β real and complex, respectively, with $|\alpha|^2 + |\beta|^2 = 1$.

[Extra credit] The Lie group $SO(3)$ is path connected. Is it simply connected?

Problem 12:

(i) Show that the standard matrix exponential

$$\exp : \mathfrak{so}(2) \to SO(2),$$

is surjective.

(ii) Show that the map

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R}),$$

is not surjective.

(iii) Additionally, prove that any matrix in $SL(2, \mathbb{R})$, close enough to the identity, is in the range of the exponential.

Notation:

$$\mathfrak{so}(2, \mathbb{R}) := \left\{ \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \mid \omega \in \mathbb{R} \right\},$$
$$\mathfrak{sl}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$